

# A GENERALIZATION OF FORELLI'S THEOREM

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ABSTRACT. The purpose of this paper is to present a generalization of Forelli's theorem. In particular, we prove an all dimensional version of the two-dimensional theorem of Chirka [1] of 2005.

## 1. INTRODUCTION

Classical Forelli's theorem ([2]; see also [5], [6]) states

**Theorem 1.1** (Forelli). *Let  $f: \mathbb{B}^n \rightarrow \mathbb{C}$  be a function. If it satisfies the following two conditions:*

(i)  $f \in C^\infty(0)$ ,

and

(ii) *for every unit vector  $v = (v_1, \dots, v_n) \in \mathbb{C}$ ,  $f(\zeta v_1, \dots, \zeta v_n)$  is holomorphic in the single complex variable  $\zeta$  with  $|\zeta| < 1$ ,*

*then  $f$  is holomorphic.*

The definition of the notation  $f \in C^\infty(0)$  in the statement is as follows: for any positive integer  $k$ , there exists a polynomial  $p_k$  such that  $f(z) - p_k(z) = o(|z|^k)$ .

After a long period of almost no results, the following two generalizations have been presented:

**Theorem 1.2** (Chirka [1]). *Let  $\{S_\tau\}$  be a foliation of a domain  $\Omega$  in a punctured ball  $\mathbb{B} \setminus 0$  in  $\mathbb{C}^2$  by holomorphic curves that are closed and smooth in  $\mathbb{B}$ , pass through the origin, and are pairwise transversal at 0. Let  $f$  be a function at  $\mathbb{B}$  such that  $f \in C^\infty(0)$  and all restrictions  $f|_{S_\tau}$  are holomorphic. Then  $f$  is holomorphic in  $\Omega$  and, if  $\Omega = \mathbb{B} \setminus 0$ , then  $f$  is holomorphic in  $\mathbb{B}$ .*

**Theorem 1.3** (Kim-Poletsky-Schmalz [4]). *If  $n \geq 1$  is an integer, and if  $f: \mathbb{B}^n \rightarrow \mathbb{C}$  is a function with  $f \in C^\infty(0)$ , which is annihilated by*

$$\bar{X} = \sum_{j=1}^n \alpha_j \bar{z}_j \frac{\partial}{\partial \bar{z}_j},$$

*where  $\alpha_1, \dots, \alpha_n$  are real numbers, then  $f$  is holomorphic on  $\mathbb{B}^n$ .*

In both versions holomorphicity along straight lines through 0 has been replaced by holomorphicity along some more general family of complex curves: in case of Theorem 1.2 they are assumed to intersect transversely. On the other hand, in the case of Theorem 1.3 the family of complex curves is generated by a holomorphic vector field and its integral curves. Thus, the two theorems complement each

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other in a sense; the general foliation considered by Chirka may not in general be generated by a contracting holomorphic vector field, whereas the leaves of foliation considered by Kim-Poletsky-Schmalz do not have to intersect mutually transversely at the origin (even after the re-parametrization of the leaves so that they intersect at the origin).

The purpose of this paper is to present Theorem 2.1, an all-dimensional version of Theorem 1.2. This in particular answers the question posed by Chirka in [1], p. 219.

## 2. A GENERALIZATION OF FORELLI'S THEOREM IN $\mathbb{C}^n$

First we define the concept of the local  $\mathcal{C}^k$  singular foliation by holomorphic curves. Let  $\Delta \subset \mathbb{C}$  the unit disc and  $S^{2n-1}$  the unit sphere in  $\mathbb{C}^n$  defined by the equation  $|z_1|^2 + \dots + |z_n|^2 = 1$ .

**Definition 2.1.** Let  $\ell$  be a positive integer. For a point  $p \in \mathbb{C}^n$ , a *local  $\mathcal{C}^\ell$  singular foliation at  $p$  by holomorphic discs* is a  $\mathcal{C}^\ell$  map  $h: \Delta \times S^{2n-1} \rightarrow \mathbb{C}^n$  satisfying the following properties:

- (1) For each  $v \in S^{2n-1}$ , the correspondence  $h(\cdot, v): z \in \Delta \rightarrow h(z, v) \in \mathbb{C}^n$  is a holomorphic embedding.
- (2)  $h(0, v) = p$  for every  $v \in S^{2n-1}$ .
- (3) The image  $h(\Delta \times S^{2n-1})$  contains an open neighborhood of  $p$  in  $\mathbb{C}^n$ .
- (4) For each  $v \in S^{2n-1}$ , there exists  $r_v > 0$  such that  $\frac{\partial h}{\partial z} \Big|_{(0,v)} = r_v v$ .
- (5)  $h(z, e^{i\theta} v) = h(e^{i\theta} z, v)$  for any  $\theta \in \mathbb{R}$  and  $z \in \Delta$ .

Throughout this paper, we shall consider the case  $\ell = 1$  only.

We shall consider, from here on, only the case when  $p$  is the origin. This singular foliation provides a parametrization of an open neighborhood of the origin in  $\mathbb{C}^n$  by  $\Delta \times \mathbb{CP}^{n-1}$ . One can always choose coordinates around the origin in  $\mathbb{C}^n$  such that any given direction  $v^0 \in S^{2n-1}$  becomes  $[1 : 0 : \dots : 0] \in \mathbb{CP}^{n-1}$  and hence a neighborhood of  $v^0$  in  $\mathbb{CP}^{n-1}$  can be identified as a neighborhood of 0 in  $\mathbb{C}^{n-1}$  with coordinates  $(\lambda^1, \dots, \lambda_{n-1}^0) := \left( \frac{v_2^0}{v_1^0}, \dots, \frac{v_n^0}{v_1^0} \right)$ . Then  $h(z, v^0)$  can be understood in this coordinate system as  $h(z, 0)$ , and in a neighborhood of the corresponding holomorphic curve, we may assume without loss of generality that it has the equation  $h(z, 0) = (z, 0)$ . Then for  $z$  and  $\lambda$ , with both  $|z|$  and  $\|\lambda\|$  sufficiently small, the curve  $z \rightarrow h(z, \lambda)$  is represented by the expression

$$\begin{cases} z &= z \\ w_1 &= k_1(z, \lambda_1, \dots, \lambda_{n-1}) \\ &\vdots \\ w_{n-1} &= k_{n-1}(z, \lambda_1, \dots, \lambda_{n-1}). \end{cases}$$

We use the short-hand notation  $w = k(z, \lambda)$ . Then  $k$  satisfies

- (i)  $k(z, \lambda)$  is  $\mathcal{C}^1$  in  $z, \lambda$  and holomorphic in  $z$ .
- (ii)  $k(0, \lambda) = 0$  for any  $\lambda$
- (iii)  $\frac{\partial k(z, \lambda)}{\partial z} \Big|_{z=0} = \lambda$  and hence  $k(z, \lambda) = z\lambda + o(|z|)$ .

Notice that we are using the standard complex coordinate system  $(z, w_1, \dots, w_{n-1})$  for  $\mathbb{C}^n$  here.

**Theorem 2.2.** *If  $h$  is a local singular foliation of a domain  $\Omega$  in  $\mathbb{C}^n$  and  $f: \Omega \rightarrow \mathbb{C}$  is a function satisfying:*

(A)  $f \in C^\infty(0) \cap C^1(\Omega)$ ; and

(B) *for every leaf  $h(\cdot, \lambda)$  the composition  $f \circ h(\cdot, \lambda)$  is a holomorphic function,*

*then  $f$  is a holomorphic function on the intersection of  $\Omega$  and some neighborhood of the origin.*

Notice that the statement of this theorem in the case of  $n = 2$  is weaker than what was presented in [1]. On the other hand, our proof here is not only valid for all dimensions, but also, even in dimension 2, somewhat more straightforward.

*Proof.* Let us use the notation  $\partial f / \partial \bar{w} := (\partial f / \partial \bar{w}_1, \dots, \partial f / \partial \bar{w}_{n-1})$ . Of course the goal here is to establish that  $\partial f / \partial \bar{w} = 0$  at  $w = 0$ .

Let  $F(z, \lambda_1, \dots, \lambda_{n-1}) = f(h(z, \lambda_1, \dots, \lambda_{n-1})) = f(z, k(z, \lambda))$ . First we prove that  $\partial F / \partial \lambda_j$ ,  $\partial F / \partial \bar{\lambda}_j$  and  $\partial k_m / \partial \lambda_j$  and  $\partial k_m / \partial \bar{\lambda}_j$  at  $\lambda = 0$  are also holomorphic in  $z$ . In fact, let  $G: \Delta \times U \rightarrow \mathbb{C}$  be a  $C^1$  function that is holomorphic with respect to  $z \in \Delta$ , like  $F, k$ . Then

$$\int G(z, \lambda) dz \wedge d\phi = 0$$

for any function in  $\phi(z) \in \mathcal{C}_0^\infty(\Delta)$ . By differentiating with respect to the parameter  $\lambda_j$  or  $\bar{\lambda}_j$  we get

$$\int_{\Delta} \frac{\partial}{\partial \lambda_j} G(z, \lambda) dz \wedge d\phi = 0, \quad \int_{\Delta} \frac{\partial}{\partial \bar{\lambda}_j} G(z, \lambda) dz \wedge d\phi = 0,$$

which shows that  $\frac{\partial G}{\partial \lambda_j}$  and  $\frac{\partial G}{\partial \bar{\lambda}_j}$  are holomorphic in  $z$ .

Since  $F$  is holomorphic in  $z$  for every  $\lambda$  and since  $F$  is  $C^1$ -smooth,  $\partial F / \partial \lambda$  and  $\partial F / \partial \bar{\lambda}$  at  $\lambda = 0$  are also holomorphic in  $z$ . By the chain rule,

$$\begin{aligned} \left. \frac{\partial F}{\partial \lambda} \right|_{\lambda=0} &= A \left. \frac{\partial f}{\partial w} \right|_{w=0} + \bar{B} \left. \frac{\partial f}{\partial \bar{w}} \right|_{w=0} \\ \left. \frac{\partial F}{\partial \bar{\lambda}} \right|_{\lambda=0} &= B \left. \frac{\partial f}{\partial w} \right|_{w=0} + \bar{A} \left. \frac{\partial f}{\partial \bar{w}} \right|_{w=0}, \end{aligned}$$

where  $\frac{\partial F}{\partial \lambda}$  denotes the column of partial derivatives  $(\frac{\partial F}{\partial \lambda_1}, \dots, \frac{\partial F}{\partial \lambda_{n-1}})$  etc. and  $A = (A_{mj})$  and  $B = (B_{mj})$  are the matrices whose entries are defined by the partial derivatives as follows:  $A_{mj} = \frac{\partial k_m}{\partial \lambda_j} \big|_{\lambda=0}$  and  $B_{mj} = \frac{\partial k_m}{\partial \bar{\lambda}_j} \big|_{\lambda=0}$ .

From  $k_m(z, \lambda) = z\lambda_m + o(|z|^2)$  we get

$$(2.1) \quad A_{mj} = \frac{\partial k_m}{\partial \lambda_j} = \delta_{mj}z + o(|z|), \quad B_{mj} = \frac{\partial k_m}{\partial \bar{\lambda}_j} = o(|z|).$$

Hence the matrix  $\tilde{A} := \frac{1}{z}A = \text{id} + o(1)$  is invertible in some neighbourhood of 0. Denote  $(\tilde{A})^{-1} = C$ . Then the entries of  $C$  are holomorphic functions. Let

$$H(z) := BC \left. \frac{\partial F}{\partial \lambda} \right|_{\lambda=0} - z \left. \frac{\partial F}{\partial \lambda} \right|_{\lambda=0} = (BC\bar{B} - z\bar{A}) \left. \frac{\partial f}{\partial \bar{w}} \right|_{w=0}.$$

Then  $H(z)$  is holomorphic on  $\Delta$ .

Now we shall prove:  $H(z) = 0, \forall z \in \Delta$ . In order to show this, we need the following lemma. Below, the notation  $\mathbb{C}[[z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_k]]$  represents the local ring of formal power series in the variables  $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_k$  at the origin. Of course, the unique maximal ideal is the set of all formal power series without the constant term.

**Lemma 2.3.** *Let  $\alpha_1, \dots, \alpha_m \in \mathbb{C}[[z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_k]]$ ,  $\psi \in \mathbb{C}[[z_1, \dots, z_n]]$  and  $\varphi_1, \dots, \varphi_m \in \mathfrak{M}$ , where  $\mathfrak{M}$  is the maximal ideal of the local ring  $\mathbb{C}[[\bar{z}_1, \dots, \bar{z}_k]]$ . Then*

$$\psi = \alpha_1 \varphi_1 + \dots + \alpha_m \varphi_m$$

*implies  $\psi = 0$ .*

*Proof.* Assume  $\psi \neq 0$  and let  $\tilde{\psi}$  be the lowest degree non-vanishing polynomial in  $\psi$ . Then

$$\tilde{\psi} = \tilde{\alpha}_1 \tilde{\varphi}_1 + \dots + \tilde{\alpha}_m \tilde{\varphi}_m,$$

where  $\tilde{\alpha}_1, \dots, \tilde{\alpha}_m$  are certain polynomials in  $z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_k$  and  $\tilde{\varphi}_1, \dots, \tilde{\varphi}_m$  are polynomials in  $\bar{z}_1, \dots, \bar{z}_k$  of positive degree. Now,  $\tilde{\psi}$  does not contain variables  $\bar{z}_1, \dots, \bar{z}_k$ , whereas each monomial in  $\tilde{\alpha}_1 \tilde{\varphi}_1 + \dots + \tilde{\alpha}_m \tilde{\varphi}_m$  does contain such variables. This contradiction shows that  $\psi = 0$ .  $\square$

The following statement is then immediate:

**Corollary 2.4.** *Assume that  $\varphi_1, \dots, \varphi_m$  and  $\alpha_1, \dots, \alpha_m$  are complex-valued functions defined on a domain  $\Omega$  in the complex plane  $\mathbb{C}$ , which enjoy the properties:*

- (a)  $\varphi_k$  has a formal Taylor expansion at  $p$ , and
- (b)  $\alpha_k$  is conjugate-holomorphic in an open neighborhood of  $p$  with  $\alpha_k(p) = 0$

*for  $k = 1, \dots, m$ . If  $\psi := \alpha_1 \varphi_1 + \dots + \alpha_m \varphi_m$  is holomorphic in  $\Omega$ , then  $\psi$  is identically zero.*

Now we return to  $H(z)$ . From (2.1) it follows that the anti-holomorphic terms  $\frac{\partial \bar{k}_m}{\partial \lambda_j}$  and  $\frac{\partial \bar{k}_m}{\partial \bar{\lambda}_j}$  vanish at  $\zeta = 0$ . Therefore, the components of  $H(z)$  have the form of a function  $\psi$  from the Corollary 2.4 with the  $\varphi$ 's being products of  $\frac{\partial f}{\partial \bar{\lambda}_j}$  and some holomorphic factors from the matrices  $A, B, C$  and the  $\alpha$ 's being products of some antiholomorphic factors from the matrices  $\bar{A}, \bar{B}$ . It follows  $H(z) \equiv 0$  and hence

$$(BC\bar{B} - z\bar{A}) \frac{\partial f}{\partial \bar{w}} \Big|_{w=0} \equiv 0.$$

Finally, even though  $BC\bar{B} - z\bar{A}$  vanishes at the origin, its determinant equals

$$\det(BC\bar{B} - z\bar{A}) = (-1)^{n-1} |z|^{2n-2} + o(|z|^{2n-2})$$

and therefore has no zeroes in some punctured neighborhood of 0. It follows that

$$\frac{\partial f}{\partial \bar{w}} \Big|_{w=0} = 0.$$

Since  $\{w = 0\}$  was an arbitrary line leaf the Cauchy-Riemann equations are satisfied transversally to each leaf. This completes the proof.  $\square$

Finally, for the global version of generalized Forelli's theorem, we recall that the following definition of global singular foliation.

**Definition 2.5.** Let  $\Omega$  be a domain in  $\mathbb{C}^n$  containing the origin. By a  $\mathcal{C}^1$  *singular foliation at 0 by holomorphic discs* we mean a  $\mathcal{C}^1$  map  $h: \Delta \times S^{2n-1} \rightarrow \Omega$  satisfying the following properties:

- (1) For each  $v \in S^{2n-1}$ , the correspondence  $h(\cdot, v): z \in \Delta \rightarrow h(z, v) \in \mathbb{C}^n$  is a holomorphic embedding.
- (2)  $h(0, v) = 0$  for every  $v \in S^{2n-1}$ .
- (3)  $h(\Delta \times S^{2n-1}) = \Omega$ .
- (4) For each  $v \in S^{2n-1}$ , there exists  $r_v > 0$  such that  $\frac{\partial h}{\partial z} \Big|_{(0,v)} = r_v v$ .
- (5)  $h(z, e^{i\theta} v) = h(e^{i\theta} z, v)$  for any  $\theta \in \mathbb{R}$  and  $z \in \Delta$ .

Then we present

**Corollary 2.6.** *Let  $S_\lambda$  be the typical leaf (i.e., a holomorphic disc) of the singular foliation as above of a domain  $\Omega \subset \mathbb{C}^n$  and  $f$  a function that is  $\mathcal{C}^\infty(0)$  and  $\mathcal{C}^1(\Omega)$  such that the restrictions  $f|_{S_\lambda}$  are holomorphic. Then  $f$  is holomorphic on  $\Omega$*

This follows from Theorem 2.2 and Chirka's curvilinear Hartogs' lemma from [1] (cf. [3]).

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